

# THE CLIFFORD TORUS AS A SELF-SHRINKER FOR THE LAGRANGIAN MEAN CURVATURE FLOW

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ABSTRACT. We provide several rigidity results for the Clifford torus in the class of compact self-shrinkers for Lagrangian mean curvature flow.

## 1. INTRODUCTION

An immersion  $\phi : M^n \rightarrow \mathbb{R}^m$  of a smooth manifold  $M$  of dimension  $n$  and codimension  $p = m - n \geq 1$  into Euclidean space is said to be a *self-shrinker* if it satisfies the quasilinear elliptic system

$$(1.1) \quad H = -\phi^\perp$$

where  $H = \text{trace } \sigma$  is the mean curvature vector of the immersion  $\phi$ , defined as the trace of the second fundamental form  $\sigma$ , and  $^\perp$  denotes the projection onto the normal bundle of  $M$ . The solutions of (1.1) not only give rise to homothetically shrinking solutions of the mean curvature flow but also they play an interesting role in the formation of type-1 singularities because it was proved by Huisken [Hu90] that solutions of the mean curvature flow forming such a singularity can be homothetically rescaled so that any resulting limiting submanifold verifies (1.1). In this way it is expected that the understanding of the singularities of the mean curvature flow will rely on the classification of self-shrinkers, but this is a hard and open problem.

There are many interesting papers (for example, [AL86], [Hu90], [Sm05], [CMi09], [LeSe10], [CaLi11], [DiXi11a], [DiXi11b], [ChZh11], [LiWe12] and [ChPe12]) about classification and rigidity of self-shrinkers for curves, hypersurfaces or arbitrary codimension, under general assumptions like compactness, completeness with polynomial volume growth, uniformly bounded geometry, proper completeness or embeddedness. In this article we are interested in rigidity results for *compact self-shrinkers in arbitrary codimension*, emphasizing the two-dimensional *Lagrangian* case. We recall that the Lagrangian constraint is preserved by the mean curvature flow.

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When  $n = 1$ , all the solutions of (1.1) are given by the Abresch-Langer curves [AL86]. Except for the straight line passing through the origin, their curvature is positive for all them and the only simple closed one is the circle.

When  $m = n + 1$  and  $n \geq 2$ , Huisken [Hu90] proved that *the only compact mean convex self-shrinker is the sphere  $\mathbb{S}^n(\sqrt{n})$  of radius  $\sqrt{n}$ .*

In higher codimension the situation becomes more complicated as the codimension increases. A natural extension of the above Huisken's result is the following theorem of Smoczyk [Sm05]: *If  $M^n$  is a compact self-shrinker in  $\mathbb{R}^m$ , then  $M^n$  is spherical if and only if  $|H| > 0$  and its principal normal vector field  $\nu = H/|H|$  is parallel in the normal bundle.* One easily observes that spherical self-shrinkers coincide with minimal submanifolds of the sphere  $\mathbb{S}^{m-1}(\sqrt{n})$  and all of them satisfy that  $|H|^2 \equiv n$ .

The simplest one (the totally geodesic  $n$ -sphere of radius  $\sqrt{n}$ ) was characterized by the following gap theorem of Cao and Li [CaLi11] (Le and Sesum [LeSe10] proved first the hypersurface case) on the squared norm of the second fundamental form: *If  $M^n$  is a compact self-shrinker in  $\mathbb{R}^m$  with  $|\sigma|^2 \leq 1$ , then  $|\sigma|^2 \equiv 1$  and  $M^n$  is the  $n$ -sphere  $\mathbb{S}^n(\sqrt{n})$  in  $\mathbb{R}^{n+1}$ .*

We observe in the above results that, in order to characterize the sphere  $\mathbb{S}^n(\sqrt{n})$ , one needs some hypothesis either on the mean curvature or on the squared norm of the second fundamental form. Very recently, Li and Wei in [LiWe12] and Cheng and Peng in [ChPe12] have obtained some rigidity results for other examples of self-shrinkers under both types of assumptions:  $|H| > 0$  and parallel principal normal vector field  $\nu = H/|H|$  and either lower and upper bounds of  $|\sigma|^2$  or constancy of  $|\sigma|^2$ .

We pay our attention to some interesting compact spherical self-shrinkers with constant squared norm of the second fundamental form:

**Example 1.1.** *For any  $n_1, n_2 \in \mathbb{N}$  such that  $n_1 + n_2 = n$ , the Clifford immersion*

$$\mathbb{S}^{n_1}(\sqrt{n_1}) \times \mathbb{S}^{n_2}(\sqrt{n_2}) \hookrightarrow \mathbb{R}^{n+2}$$

*is a compact self-shrinker in  $\mathbb{R}^{n+2}$  with  $|\sigma|^2 \equiv 2$ .*

**Example 1.2.** *The product of  $n$ -circles*

$$\mathbb{S}^1 \times \overset{n)}{\dots} \times \mathbb{S}^1 \hookrightarrow \mathbb{R}^{2n}$$

*is a compact flat self-shrinker with  $|\sigma|^2 \equiv n$  that is Lagrangian in  $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ .*

**Example 1.3.** *The immersion*

$$\mathbb{S}^1 \times \mathbb{S}^{n-1} \rightarrow \mathbb{C}^n \equiv \mathbb{R}^{2n}, \quad (e^{it}, (x_1, \dots, x_n)) \mapsto \sqrt{n} e^{it} (x_1, \dots, x_n)$$

*is a compact self-shrinker with  $|\sigma|^2 \equiv \frac{3n-2}{n} \in [2, 3)$  that is Lagrangian in  $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ .*

By translating the well-known results of Simon [Si68], Lawson [La69] and Chern-Do Carmo-Kobayashi [CdCK78] about intrinsic rigidity for minimal submanifolds in the unit sphere and using some simple observations

of [CaLi11], we arrive at the following gap result for compact self-shrinkers of codimension  $p \geq 1$ .

**Theorem A.** *Let  $\phi : M^n \rightarrow \mathbb{R}^{n+p}$  be a compact self-shrinker such that  $|H|^2$  is constant or  $|H|^2 \leq n$  or  $|H|^2 \geq n$ . If*

$$(1.2) \quad |\sigma|^2 \leq \frac{3p-4}{2p-3}$$

*then:*

- (1) *either  $|\sigma|^2 \equiv 1$  and  $M$  is  $\mathbb{S}^n(\sqrt{n})$  in  $\mathbb{R}^{n+1}$  (i.e.  $p = 1$ ),*
- (2) *or  $|\sigma|^2 \equiv \frac{3p-4}{2p-3}$  and  $M$  is*
  - (a) *either  $\mathbb{S}^{n_1}(\sqrt{n_1}) \times \mathbb{S}^{n_2}(\sqrt{n_2})$ ,  $n_1 + n_2 = n$ , (with  $|\sigma|^2 \equiv 2$ ) in  $\mathbb{R}^{n+2}$  (i.e.  $p = 2$ ),*
  - (b) *or the Veronese immersion (with  $|\sigma|^2 \equiv 5/3$ ) of  $\mathbb{S}^2(\sqrt{6})$  in  $\mathbb{R}^5$  (i.e.  $n = 2, p = 3$ ).*

We remark that our hypothesis on  $H$  of Theorem A in the compact case is weaker than  $|H| > 0$  and parallel principal normal vector field  $\nu = H/|H|$ . Our result generalizes Theorem 1.2 of [LiWe12] in the compact case.

If the dimension and the codimension of the submanifold  $M$  coincide (what happens, for example, when  $M$  is Lagrangian), as a first immediate consequence of Theorem A we obtain the following surprising characterization of the Clifford torus  $\mathbb{S}^1 \times \mathbb{S}^1$ .

**Corollary A.** *Let  $\phi : M^n \rightarrow \mathbb{R}^{2n}$  be a compact self-shrinker with codimension  $n \geq 2$  such that  $|H|^2$  is constant or  $|H|^2 \leq n$  or  $|H|^2 \geq n$ . If*

$$(1.3) \quad |\sigma|^2 \leq \frac{3n-4}{2n-3}$$

*then  $n = 2$ ,  $|\sigma|^2 \equiv 2$  and  $M$  is the Clifford torus  $\mathbb{S}^1 \times \mathbb{S}^1$  in  $\mathbb{R}^4$ .*

It is quite remarkable that the above three examples (Example 1.1, 1.2 and 1.3) coincide when  $n = 2$  providing precisely the Clifford torus  $\mathbb{S}^1 \times \mathbb{S}^1$  in  $\mathbb{R}^4$ . In fact, we remark that Example 1.2 and Example 1.3 satisfy the hypothesis of Corollary A only when  $n = 2$ . So it can be expectable some other rigidity results in this setting for this regular example. But the Clifford torus  $\mathbb{S}^1 \times \mathbb{S}^1$  is not isolated in the class of compact self-shrinkers in Euclidean 4-space. In fact, it belongs to four different families of infinitely many self-shrinkers of genus one, that we will study in section 3:

- (i) *Abresch-Langer tori*, product of two Abresch-Langer curves [AL86];
- (ii) *Anciaux tori*, defined by considering the case  $n = 2$  in Theorem 1 of [An06];
- (iii) *Lee-Wang tori*, defined by considering the case  $n = 2$  in Proposition 2.1 of [LW10] and described explicitly in Proposition 3 of [CL10];
- (iv) *Lawson tori*, described in Theorem 3 of [La70].

Thus, rigidity theorems in the family of compact (Lagrangian) self-shrinkers in  $\mathbb{R}^4$  are welcome and the Clifford torus is the natural candidate for this

purpose. This type of results may be useful to try to get some progress related with the open Question 7.4 of [Ne10]. Our contribution to this problem consists of three different new characterizations of the Clifford torus, assuming in the Lagrangian setting only one type of assumptions: either on  $H$  or on  $|\sigma|^2$ .

**Theorem 1.1.** *Let  $\phi : M^2 \rightarrow \mathbb{R}^4$  be a compact Lagrangian self-shrinker. If  $|H|^2$  is constant or  $|H|^2 \leq 2$  or  $|H|^2 \geq 2$ , then  $M^2$  is the Clifford torus  $\mathbb{S}^1 \times \mathbb{S}^1$ .*

As a consequence of this result we get that *the Clifford torus is the only compact Lagrangian spherical self-shrinker in  $\mathbb{R}^4$* . By considering all the previous results, it seems to be interesting the study of compact self-shrinkers with  $|\sigma|^2 \leq 2$ . Since there do not exist Lagrangian self-shrinking spheres (see Theorem 2.1), making a subtle combination of Gauss-Bonnet Theorem with the formula (2.3) that expresses the Willmore functional as a integer multiple of the area of a compact self-shrinker, we get the following result.

**Theorem 1.2.** *Let  $\phi : M^2 \rightarrow \mathbb{R}^4$  be a compact orientable Lagrangian self-shrinker. If  $|\sigma|^2 \leq 2$ , then  $|\sigma|^2 \equiv 2$  and  $M$  is a torus. If, in addition, the Gauss curvature  $K$  of  $M$  is non-negative or non-positive, then  $M^2$  is the Clifford torus  $\mathbb{S}^1 \times \mathbb{S}^1$ .*

In [CL10] the authors classified all Hamiltonian stationary Lagrangian surfaces in complex Euclidean plane which are self-similar solutions of the mean curvature flow. The Hamiltonian stationary condition is equivalent to the vanishing of the divergence of the tangent vector field  $JH$ , being  $J$  the standard complex structure of  $\mathbb{C}^2$ . Based on the above mentioned classification, we finally deduce:

**Theorem 1.3.** *Let  $\phi : M^2 \rightarrow \mathbb{R}^4$  be a compact self-shrinker. If  $\phi$  is a Hamiltonian stationary Lagrangian embedding, then  $M^2$  is the Clifford torus  $\mathbb{S}^1 \times \mathbb{S}^1$ .*

After analyzing the different hypothesis in our characterizations of the Clifford torus with the four families of self-shrinking tori cited above, we conjecture that *the Clifford torus is the only compact Lagrangian self-shrinker in  $\mathbb{R}^4$  with  $|\sigma|^2 \leq 2$* .

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## 2. PRELIMINARIES

Let  $\phi : M^n \rightarrow \mathbb{R}^m$  be an isometric immersion of an  $n$ -dimensional submanifold in Euclidean  $m$ -space. The mean curvature vector  $H$  of  $\phi$  is given by  $H = \text{trace } \sigma$ , where  $\sigma$  denotes the second fundamental form of  $\phi$ . A submanifold  $M$  in  $\mathbb{R}^m$  is called a *self-shrinker* if

$$(2.1) \quad H = -\phi^\perp$$

where  $^\perp$  stands for the projection onto the normal bundle of  $M$ . If  $^\top$  denotes projection onto the tangent bundle, it is easy to check that  $\phi^\top = \frac{1}{2}\nabla|\phi|^2$ , where  $\nabla$  means gradient with respect to the induced metric on  $M$ .

Let  $\phi : M^n \rightarrow \mathbb{R}^m$  be a self-shrinker. Using (2.1), we get the following formula for the Laplacian of the squared norm of  $\phi$ :

$$(2.2) \quad \Delta|\phi|^2 = 2(n - |H|^2)$$

In particular, when  $M$  is compact, we obtain an interesting relationship between the Willmore functional of  $\phi$  and the area of  $M$ :

$$(2.3) \quad \int_M |H|^2 d\mu = n \text{Area}(M)$$

We can find many examples (see Examples 1.1, 1.2 and 1.3 in section 1) in the class of *spherical self-shrinkers*:

Consider that  $\phi : M^n \rightarrow \mathbb{S}^{m-1}(R) \subset \mathbb{R}^m$  is a spherical immersion with second fundamental form  $\hat{\sigma}$  and mean curvature vector  $\hat{H}$ . Then  $\phi$  is a self-shrinker if and only if  $\hat{H} = 0$  and  $R = \sqrt{n}$ , that is,  $M$  is a minimal submanifold in the  $(m-1)$ -sphere of radius  $\sqrt{n}$ . In this case,  $H = -\phi$  and so,  $|H|^2 = |\phi|^2 = n$  and, in addition, it satisfies

$$(2.4) \quad |\sigma|^2 = 1 + |\hat{\sigma}|^2$$

We consider now a special case of codimension  $n$ : the Lagrangian submanifolds, recalling that the Lagrangian constraint is preserved by the mean curvature flow. An immersion  $\phi : M^n \rightarrow \mathbb{R}^{2n} \equiv \mathbb{C}^n$  is said to be Lagrangian if the restriction to  $M$  of the Kaehler two-form  $\omega(\cdot, \cdot) = \langle J\cdot, \cdot \rangle$  of  $\mathbb{C}^n$  vanishes. Here  $J$  is the complex structure on  $\mathbb{C}^n$  that defines a bundle isomorphism between the tangent and the normal bundle of  $\phi$ . In particular,  $\sigma(v, w) = JA_J w$ , where  $A$  is the shape operator, and so the trilinear form  $C(\cdot, \cdot, \cdot) = \langle \sigma(\cdot, \cdot), J\cdot \rangle$  is fully symmetric. On the other hand,

$$(2.5) \quad H = J\nabla\beta,$$

where  $\beta : M \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  is called the Lagrangian angle map of  $\phi$ . In general  $\beta$  is a multivalued function; nevertheless  $\alpha_H = -d\beta = \langle JH, \cdot \rangle$  is a well defined closed 1-form on  $M$  and its cohomology class  $[\alpha_H]$  is called the Maslov class of  $\phi$ .

Suppose now that  $\phi : M^n \rightarrow \mathbb{R}^{2n} \equiv \mathbb{C}^n$  is a *Lagrangian self-shrinker*. Derivating  $\phi = \phi^\top - H$  in the direction of any tangent vector  $v \in TM$  and separating tangent and normal components, we get that

$$(2.6) \quad A_H v = v - \nabla_v \phi^\top$$

where  $A_H$  is the Weingarten endomorphism associated to  $H$  and  $\nabla$  is the Levi-Civita connection of  $M$ , and

$$(2.7) \quad \nabla_v^\perp H = \sigma(v, \phi^\top)$$

where  $\nabla^\perp$  is the connection of the normal bundle. Using (2.7) we obtain that

$$(2.8) \quad \operatorname{div} JH = \langle JH, \phi^\top \rangle$$

where  $\operatorname{div}$  denotes the divergence operator.

We finish this section with an interesting result, first proved by Smoczyk, about the non existence of compact orientable Lagrangian self-shrinkers with trivial Maslov class.

**Theorem 2.1.** *Let  $\phi : M^n \rightarrow \mathbb{C}^n$  be a Lagrangian self-shrinker. If  $M$  is compact orientable, then  $[\alpha_H] \neq 0$ .*

*Proof.* Assume that  $[\alpha_H] = 0$ . Then there exists a globally defined Lagrangian angle  $\beta$  such that  $\alpha_H = -d\beta$ . Taking into account that  $\Delta\beta = -\operatorname{div} JH$ , from (2.8) we have that  $\beta$  satisfies the elliptic linear equation  $\Delta\beta = \frac{1}{2}\langle \nabla\beta, \nabla|\phi|^2 \rangle$ . Then the maximum principle says that  $\beta$  must be constant and thus  $H \equiv 0$ . Since there are no compact minimal submanifolds in Euclidean space, this is a contradiction.  $\square$

As a consequence of Theorem 2.1, *there do not exist Lagrangian self-shrinkers with the topology of a sphere.*

### 3. SELF-SHRINKING TORI IN EUCLIDEAN 4-SPACE

In this section we collect the main geometric properties of four families of self-shrinking tori all them including the Clifford torus.

**3.1. Abresch-Langer tori.** The Abresch-Langer tori are defined simply as the product  $\Gamma_1 \times \Gamma_2$  of two closed Abresch-Langer curves [AL86].

The curvature vector of such a curve satisfies

$$(3.1) \quad \vec{\kappa}_{\Gamma_i} = -\Gamma_i^\perp \Leftrightarrow \kappa_{\Gamma_i} = \langle \Gamma_i', i\Gamma_i \rangle, \quad i = 1, 2$$

where  $'$  denotes derivative respect to the arclength parameter. The only simple closed one is the unit circle and the product of two unit circles gives obviously the Clifford torus  $\mathbb{S}^1 \times \mathbb{S}^1$ . Equation (3.1) admits a countable family of closed noncircular solutions, which is parametrized by relatively prime numbers  $p_i$  and  $q_i$  such that  $p_i/q_i \in (1/2, 1/\sqrt{2})$ ,  $i = 1, 2$ . Moreover, if  $r_i = |\Gamma_i|$ ,  $i = 1, 2$ , then one can deduce from (3.1) that  $\kappa_{\Gamma_i} = \rho_i e^{r_i^2/2}$  with  $r_i^2(1 - r_i'^2)e^{-r_i^2} = \rho_i^2$ , being  $\rho_i > 0$  a positive constant depending on  $p_i$  and  $q_i$ ,  $i = 1, 2$ .

The Abresch-Langer tori are flat Lagrangian tori whose second fundamental form and mean curvature vector satisfy

$$(3.2) \quad |\sigma|^2 = |H|^2 = \kappa_{\Gamma_1}^2 + \kappa_{\Gamma_2}^2 = \rho_1^2 e^{|\Gamma_1|^2} + \rho_2^2 e^{|\Gamma_2|^2} > 0$$

The only embedded Abresch-Langer torus is the Clifford torus.

**3.2. Anciaux tori.** Using the case  $n = 2$  in Theorem 1 of [An06], we can define the Anciaux tori by the immersions  $\phi_{p,q} : I \times \mathbb{R} \rightarrow \mathbb{C}^2$ , parametrized by relatively prime numbers  $p$  and  $q$  such that  $p/q \in (1/4, 1/2)$ , given by

$$(3.3) \quad \phi_{p,q}(t, s) = \gamma_{p,q}(t)(\cos s, \sin s)$$

where  $\gamma = \gamma_{p,q}(t)$ ,  $t \in I \subset \mathbb{R}$ , is a closed curve such that its curvature satisfies the equation

$$(3.4) \quad \kappa_\gamma = \frac{\langle \gamma', i\gamma \rangle}{|\gamma|^2} (|\gamma|^2 - 1)$$

where  $'$  denotes derivative respect to the arclength parameter. It is clear that  $\gamma(t) = \sqrt{2}e^{it/\sqrt{2}}$  satisfies (3.4) providing the Clifford torus.

Following [An06] we deduce from (3.4) that

$$(3.5) \quad \kappa_\gamma = \frac{E e^{r^2/2}(r^2 - 1)}{r^3}, \quad r = |\gamma|, \quad r^4(1 - r'^2)e^{-r^2} = E^2$$

being  $E > 0$  a positive constant depending on  $p$  and  $q$ .

The squared norm of the mean curvature vector of an Anciaux torus is given by

$$(3.6) \quad |H_{p,q}|^2 = \frac{E^2 e^{r^2}}{r^2}$$

and the squared norm of the second fundamental form of an Anciaux torus is given by

$$(3.7) \quad |\sigma_{p,q}|^2 = \frac{E^2 e^{r^2}}{r^6} (r^4 - 2r^2 + 4)$$

Every Anciaux torus is Lagrangian but the only embedded one is the Clifford torus by Theorem 3 in [An06].

**3.3. Lee-Wang tori.** From [CL10] and [LW10], we define the Lee-Wang tori  $\mathcal{T}_{m,n}$  by the doubly-periodic immersions  $\Psi_{m,n} : \mathbb{R}^2 \rightarrow \mathbb{C}^2$ ,  $m, n \in \mathbb{N}$ ,  $(m, n) = 1$ ,  $m \leq n$ , given by

$$(3.8) \quad \Psi_{m,n}(s, t) = \sqrt{m+n} \left( \frac{1}{\sqrt{n}} \cos s e^{i\sqrt{\frac{n}{m}}t}, \frac{1}{\sqrt{m}} \sin s e^{i\sqrt{\frac{m}{n}}t} \right)$$

The Clifford torus corresponds to  $\mathcal{T}_{1,1}$ , since  $\Psi_{1,1}(s, t) = \sqrt{2}e^{it}(\cos s, \sin s)$ .

The Lee-Wang tori are Hamiltonian stationary Lagrangian tori satisfying

$$(3.9) \quad \frac{m+n}{n} \leq |\Psi_{m,n}|^2 = \frac{m+n}{mn} (m \cos^2 s + n \sin^2 s) \leq \frac{m+n}{m}$$

After a straightforward computation, the squared norm of the mean curvature vector of a Lee-Wang torus is given by

$$(3.10) \quad 1 < \frac{m+n}{n} \leq |H_{m,n}|^2 = \frac{m+n}{n \cos^2 s + m \sin^2 s} \leq \frac{m+n}{m}$$

and the squared norm of the second fundamental form and the Gauss curvature of a Lee-Wang torus verify

$$(3.11) \quad \frac{3m^2 + n^2}{n(m+n)} \leq |\sigma_{m,n}|^2 \leq \frac{m^2 + 3n^2}{m(m+n)}$$

and

$$(3.12) \quad -\frac{n(n-m)}{m(m+n)} \leq K_{m,n} \leq \frac{m(n-m)}{n(m+n)}$$

The only embedded Lee-Wang torus is the Clifford torus by Proposition 3 in [CL10].

**3.4. Lawson tori.** We define (cf. [La70]) the Lawson tori  $\mathcal{T}_\alpha$  by the doubly-periodic immersions  $\Phi_\alpha : \mathbb{R}^2 \rightarrow \mathbb{C}^2$ ,  $\alpha \in \mathbb{Q}$ ,  $\alpha \geq 1$ , given by

$$(3.13) \quad \Phi_\alpha(x, y) = \sqrt{2} (\cos x e^{i\alpha y}, \sin x e^{iy})$$

The Clifford torus corresponds to  $\mathcal{T}_1$ , since  $\Phi_1(x, y) = \sqrt{2} e^{iy} (\cos x, \sin x)$ .

The Lawson tori are spherical tori whose squared norm of the second fundamental form is given by

$$(3.14) \quad 1 + \frac{1}{\alpha^2} \leq |\sigma_\alpha|^2 = 1 + \frac{\alpha^2}{(\alpha^2 \cos^2 x + \sin^2 x)^2} \leq 1 + \alpha^2$$

The Gauss curvature of a Lawson torus verifies

$$(3.15) \quad 1 - \alpha^2 \leq K_\alpha \leq 1 - \frac{1}{\alpha^2}$$

The only embedded Lawson torus is the Clifford torus, which is also the only Lagrangian in this family.

#### 4. PROOF OF THE RESULTS

In this section we prove the results stated in section 1.

*Proof of Theorem A.*

We first integrate (2.2) using that  $M$  is compact, obtaining  $0 = \int_M (n - |H|^2) d\mu$ . By the hypothesis on the mean curvature  $H$  of  $\phi$  we conclude  $|H|^2 \equiv n$ . Using (2.2) again, we deduce that  $|\phi|^2$  is harmonic, so it must be constant. Using section 2, we have that  $\phi$  is a minimal submanifold in the sphere  $\mathbb{S}^{n+p-1}(\sqrt{n})$ .

In addition, from (2.4) and (1.2), we obtain that its second fundamental form  $\hat{\sigma}$  satisfies

$$(4.1) \quad |\hat{\sigma}|^2 \leq \frac{p-1}{2p-3}.$$

We now recall the well known results of [Si68], [La69] and [CdCK78], about intrinsic rigidity for minimal submanifolds in the unit sphere, which can be summarized in the following way:

*If  $M^n$  is a compact minimal submanifold of  $\mathbb{S}^{n+q}$  with second fundamental form  $\hat{\sigma}$  such that  $|\hat{\sigma}|^2 \leq n/(2-1/q)$ , then either  $|\hat{\sigma}|^2 \equiv 0$  and  $M^n$  is  $\mathbb{S}^n$ , or*



$|\hat{\sigma}|^2 \equiv n/(2-1/q)$  and  $M^n$  is  $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$  in  $\mathbb{S}^{n+1}$ ,  $1 \leq k \leq n-1$ , or the Veronese immersion of  $\mathbb{S}^2(\sqrt{3})$  in  $\mathbb{S}^4$ .

Up to a dilation in  $\mathbb{R}^{n+q+1} \supset \mathbb{S}^{n+q}$  of ratio  $\sqrt{n}$ , we can rewrite it as follows:

If  $M^n$  is a compact minimal submanifold of  $\mathbb{S}^{n+q}(\sqrt{n})$  with second fundamental form  $\hat{\sigma}$  such that  $|\hat{\sigma}|^2 \leq q/(2q-1)$ , then either  $|\hat{\sigma}|^2 \equiv 0$  and  $M^n$  is  $\mathbb{S}^n(\sqrt{n})$ , or  $|\hat{\sigma}|^2 \equiv q/(2q-1)$  and  $M^n$  is  $\mathbb{S}^k(\sqrt{k}) \times \mathbb{S}^{n-k}(\sqrt{n-k})$  in  $\mathbb{S}^{n+1}(\sqrt{n})$ ,  $1 \leq k \leq n-1$ , or the Veronese immersion of  $\mathbb{S}^2(\sqrt{6})$  in  $\mathbb{S}^4(\sqrt{2})$ .

Taking  $q = p - 1$ , thanks to (4.1), we make use of the above result to finish the proof of Theorem A.

*Proof of Theorem 1.1.*

The same argument of the first part of the proof of Theorem A implies that  $\phi$  is spherical. So we know from section 2 that  $|\phi|^2 = |H|^2 \equiv 2$  and Theorem 1.1 in [Sm05] says that  $\nabla^\perp H = 0$ . Since  $\phi$  is Lagrangian we have that  $JH$  is a non-null parallel tangent vector field on  $M$ . We use then Theorem 3 in [Ur87] to deduce that  $M$  must be necessarily a standard torus, product of two circles. Using finally that  $\phi$  is a self-shrinker,  $M$  only can be the Clifford torus.

*Proof of Theorem 1.2.*

We use Gauss-Bonnet Theorem in the Gauss equation of  $\phi$

$$(4.2) \quad 2K = |H|^2 - |\sigma|^2$$

obtaining

$$(4.3) \quad 8\pi(1 - \text{gen}(M)) = 2 \int_M K d\mu = \int_M (|H|^2 - |\sigma|^2) d\mu = \int_M (2 - |\sigma|^2) d\mu,$$

the last equality thanks to (2.3).

Theorem 2.1 implies that  $M$  can not be a sphere. Hence the hypothesis  $|\sigma|^2 \leq 2$  in (4.3) says that  $M$  is a torus with  $|\sigma|^2 \equiv 2$ . If, in addition,  $K \geq 0$  or  $K \leq 0$ , we deduce from (4.2) that  $|H|^2 \geq 2$  or  $|H|^2 \leq 2$ . Then Theorem 1.1 gives that  $M$  is the Clifford torus.

*Proof of Theorem 1.3.*

In Corollary 1 of [CL10], the authors proved that the Lee-Wang tori  $\mathcal{T}_{m,n}$  are the only compact orientable Hamiltonian stationary Lagrangian self-shrinkers. Since the Clifford torus  $\mathcal{T}_{1,1}$  is the only embedded in this family, we finish the proof using that a Klein bottle does not admit a Lagrangian embedding in  $\mathbb{C}^2$  (see [Nm09]).

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